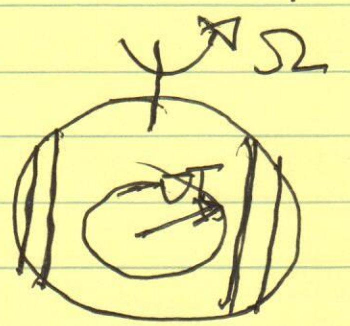
⇒ Further develop:

⇒ Rotating Systems

- Many astrophysical systems:

  ⇒ convection + rotation



How do they interact?

- Key point: rotation changes the $\begin{cases} \text{freezing-in law} \\ \text{vorticity dynamics.} \end{cases}$

Recall:

$$\partial_t \underline{\omega} - \nu \nabla^2 \underline{\omega} = \nabla \times \underline{v} \times \underline{\omega}$$

induction eqn. for vorticity

akin:
$$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = \eta \underline{J}$$

$$\nabla \times \underline{E} = -\frac{1}{c} \frac{\partial \underline{B}}{\partial t}$$

then

$$\frac{\partial \underline{B}}{\partial t} - \eta \nabla^2 \underline{B} = \underline{\nabla} \times \underline{V} \times \underline{B}$$

Compressible: $\underline{\omega}/\rho$, $\underline{B}/\rho$ frozen in

hereafter incompressible. what of anelastic?

Now, $\partial_t \underline{\omega} + \underline{V}\cdot\underline{\nabla}\,\underline{\omega} - \nu\nabla^2\underline{\omega} = \underline{\omega}\cdot\underline{\nabla}\,\underline{V}$

$$\frac{d\underline{\omega}}{dt} = \underline{\omega}\cdot\underline{\nabla}\,\underline{V}$$ → note structure.

Consider 2 test particles at $\underline{r}_1$, $\underline{r}_2$

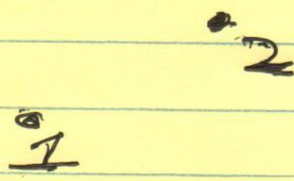
"Frozen into" flow:

$$\frac{d\underline{u}}{dt} = -\gamma\left(\underline{u} - \underline{V}(\underline{x},t)\right) + \tilde{\underline{f}}.$$

so

$$\frac{d\underline{r}_1}{dt} = \underline{V}(\underline{r}_1, t)$$

$$\frac{d\underline{r}_2}{dt} = \underline{V}(\underline{r}_2, t)$$



so let $\delta\underline{l} = \underline{r}_2 - \underline{r}_1$

think of as denoting filament

$$\frac{d}{dt}\delta\underline{l} = \underline{v}\left(\underline{r}_+ + \frac{\delta\underline{l}}{2}, t\right) - \underline{v}\left(\underline{r}_+ - \frac{\delta\underline{l}}{2}, t\right)$$

$$\cong \delta\underline{l}\cdot\underline{\nabla}\,\underline{v}$$

so

$$\boxed{\frac{d}{dt}\delta\underline{l} = \delta\underline{l}\cdot\underline{\nabla}\,\underline{v}}$$

eqn. for frozen in vector filament

→ $d\underline{l}$ follows the flow and is stretched by it

→ $d\underline{l}$ is vector field "frozen onto" the flow.

Since: $\dfrac{d\underline{\omega}}{dt} = \underline{\omega}\cdot\underline{\nabla}\,\underline{v}$

has same form as $\delta\underline{l}$ equation, we say vorticity 'frozen onto' the flow. Vortex filaments follow, and are stretched by, the flow.

Obviously: $\underline{\text{freezing-in} \leftrightarrow \text{Kelvin's Thm}}$

↓ local, field (vector) condition

↓ integral (oriented) scalar.

Now, in rotating fluid:

$$\Omega = \Omega\hat{z}$$

$\underset{\text{frame}}{\underline{V}} \to \underline{V} + \underline{\Omega}\times\underline{r}$, etc.

so, for $\underline{\nabla}\cdot\underline{V} = 0$,

$$\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V}\right) = -\underline{\nabla}\left(\frac{P}{\rho} - \underbrace{\frac{\Omega^2 r^2}{2}}_{\text{centrifugal force (absorb into pressure)}}\right) + \underbrace{\underline{V}\times 2\underline{\Omega}}_{\text{coriolis force.}}$$

$$\underline{V}\cdot\underline{\nabla}\,\underline{V} = \underline{\nabla}\left(V^2/2\right) - \underline{V}\times\underline{\omega}$$

$\Rightarrow$

$$\frac{\partial \underline{V}}{\partial t} = -\underline{\nabla}\left(\frac{P}{\rho} + \frac{V^2}{2} - \Omega^2 r^2\right) + \underline{V}\times\underline{\omega} + \underline{V}\times 2\underline{\Omega}$$

relative vorticity; planetary.

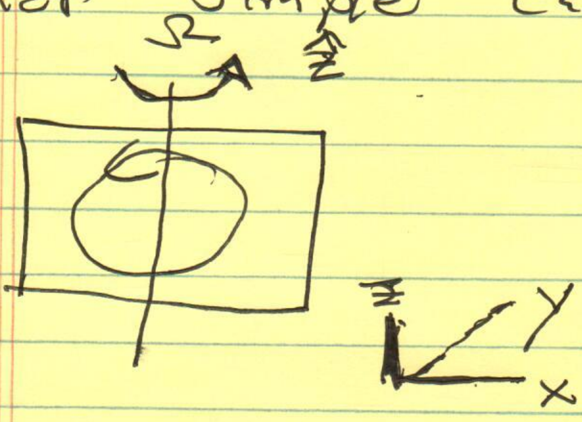
so

$$\frac{\partial}{\partial t}\left(\underline{\omega} + 2\underline{\Omega}\right) = \underline{\nabla}\times\underline{V}\times\left(\underline{\omega} + \underbrace{2\underline{\Omega}}_{\text{mean vorticity}}\right) + \nu\nabla^2\underline{\omega}$$

$\therefore$ – $\underline{\omega} + 2\underline{\Omega}$ frozen in

– $\int d\underline{a}\cdot\left(\underline{\omega} + 2\underline{\Omega}\right)$ conserved to ν.

So, for simple case:



later consider tilts, etc.

→ GFD

$\therefore$

$$\frac{\partial \underline{\omega}}{\partial t} + \underline{v} \cdot \nabla \left(\underline{\omega} + 2\underline{\Omega} \right) - \nu \nabla^2 \underline{\omega}$$
$$= \left(2\underline{\Omega} + \underline{\omega} \right) \cdot \nabla \underline{v}$$

$$\frac{d\underline{\omega}}{dt} - \nu \nabla^2 \underline{\omega} = \left(2\underline{\Omega} + \underline{\omega} \right) \cdot \underline{\nabla}\, \underline{v}$$

$$= 2\Omega \frac{\partial}{\partial z} \underline{v} + \underline{\omega} \cdot \underline{\nabla} \underline{v}$$

For $\Omega \gg |\underline{\omega}|, |\underline{\nabla} v|$;
+ mean vorticity ;
i.e. system rotation strong compared to relative vorticity,

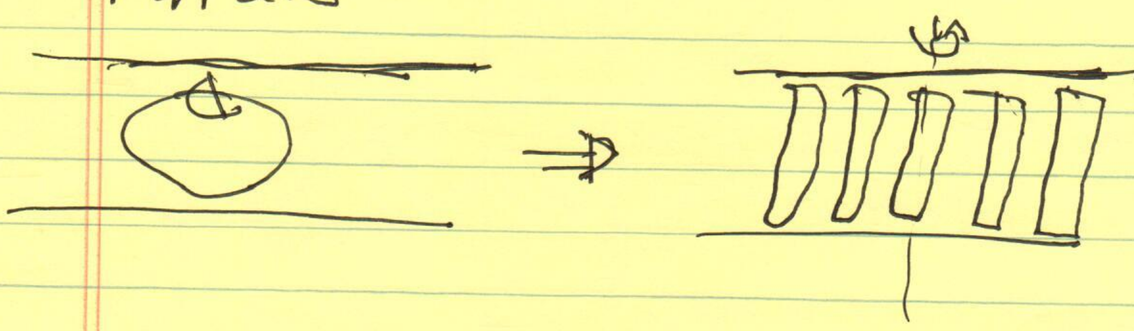
- motions cannot vary in direction of $\underline{\Omega}$
- all steady motions in rotating, inviscid flow, are necessarily two dimensional..

$$\boxed{2\Omega \frac{\partial}{\partial z} \underline{v} \cong 0}$$

→ **Taylor–Proudman Theorem (1921, 1916)**

⇒ Rapid rotation "two dimensionalizes" the flow.

⇒ cells organize into "Taylor columns", "Proudman Pillars" ⇒ i.e. shift in cartoon.



⇒ viscous boundary layer top/bottom νk^2 → Ekman Layer

(fluid rotating, tank not)

may enter.

Next: Consider:
- waves in rotating fluids
- Convection in rotating systems.